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Functional characterization of generalized Langevin equations

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Abstract

We present an exact functional formalism to deal with linear Langevin equations with arbitrary memory kernels and driven by an arbitrary noise structure characterized through its characteristic functional. No other hypothesis is assumed over the noise, neither do we use the fluctuation–dissipation theorem. We find that the characteristic functional of the linear process can be expressed in terms of noise functional and the Green function of the deterministic (memory-like) dissipative dynamics. This yields a procedure for calculating the full Kolmogorov hierarchy of the non-Markov process. As examples, we have characterized through the 1-time probability a noise-induced interplay between the dissipative dynamics and the structure of different noises. Conditions that lead to non-Gaussian statistics and distributions with long tails are analysed. The introduction of arbitrary fluctuations in fractional Langevin equations has also been pointed out.

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1. Introduction

Noise is a basic ingredient of many types of models in physics, mathematics, economics, as well as in engineering. While in each area of research the fluctuations have very different origins, in many cases the evolution equation governing the system of interest can be approximated by a suitable stochastic differential equation.

In general, the driving forces may be any source of fluctuations and the system can be characterized by a given potential. The most popular of those stochastic differential equations are those driven by white Gaussian fluctuations, where the problem can immediately be reduced to the well-known Fokker–Planck dynamics [1–3]. If the fluctuations are not Gaussian we are faced with a problem that is hard to solve, but among the different types of fluctuations the

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so-called dichotomic noise is a good candidate to study, because in general for any potential some conclusions can be drawn [4–8]. If the fluctuations are neither Gaussiann or dichotomic noise, in general it is not possible to solve the problem for an arbitrary potential.

In many situations, the technical complications of the model can be reduced by studying the system in a linear approximation, i.e., around the fixed points of the dissipative dynamics. This leads to the study of linear stochastic differential equations with arbitrary noises. Besides the simplicity of this kind of equations they have been the subject of extensive theoretical investigation [9–25], and they also provide non-trivial models for the study of many different mechanisms of relaxation in physics, biology and other research areas.

From the above considerations, it is clear that the usefulness of a linear Langevin equation arises from the possibility of working with different kinds of noise structures. Therefore, one is faced with the characterization, in general, of non-Markov processes. These processes can only be completely characterized through the whole Kolmogorov hierarchy of the stochastic process [2], i.e., the knowledge of any n-joint probability, or equivalently any n-time moment and/or cumulant.

In [26, 27], by using a functional technique, we have been able to characterize arbitrary linear Langevin equations with local dissipation, finding therefore a procedure to calculate the whole Kolmogorov hierarchy of the process and any n-time moment. Using this previous experience, in this paper we generalize our functional technique to tackle the more general situation where the dissipative term is non-local in time and the noise is also arbitrary; this is what we call a *generalized Langevin equation*, i.e., a linear Langevin equation with arbitrary memory and driven by any noise structure,

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = -\int_0^t \mathrm{d}t'\,\Phi(t-t')u(t') + \xi(t) \qquad u \in (-\infty, +\infty) \tag{1.1}$$

where $\xi(t)$, the fluctuation term (i.e., the external noise) is characterized by its associated functional, and $\Phi(t)$ is an arbitrary memory kernel.

This type of generalized Langevin equation arises quite naturally (considering Gaussian fluctuations) in the context of the Zwanzig–Mori projector operator technique [28, 29]; in this case the fluctuation–dissipation theorem is required [30, 31], which imposes that $\Phi(t) = \langle \xi(t+\tau)\xi(\tau)\rangle / kT$. Therefore the dissipative memory must be consistent with the structure of the correlation of the Gaussian fluctuations [30]. Nevertheless, if the system is far away from equilibrium, the fluctuation–dissipation theorem does not apply and in general the Gaussian assumption is not a good candidate to describe the fluctuations of the system. Therefore, in a general situation, both the kernel and the noise properties must be considered independent elements whose interplay will determine the full stochastic dynamics of the process $u(t)$. In what follows, we will be interested in characterizing this noise-induced interplay by assuming different kinds of noises and memory kernels, both in the transient as in the long-time regime.

The paper is organized as follows. In section 2, after a short review of the functional method, we obtain the characteristic functional for a vectorial Langevin equation with memory. In section 3 we apply this result to different situations. First we analyse the case of stable noises; next, we analyse the stochastic dynamics induced by an exponential memory kernel in a process driven by two different fluctuating structures: radioactive and Poisson noises. Then the interplay between the kernel and the noise properties is emphasized. The necessary properties that guarantee a stationary distribution with a long tail are presented. As an example, we introduce the Abel noise that induces this asymptotic behaviour. The characterization of a fractional Langevin equation with arbitrary noise is also presented. The interplay between the fractional property of the differential operator and Lévy noise is analysed. In section 4 we give the conclusions.

2. Functional characterization of arbitrary linear Langevin equations

2.1. The characteristic functional method

In our previous papers, we have presented a complete characterization of Langevin equations with local dissipation by means of the characteristic functional of the stochastic process $u(t)$ ($t \in (0, \infty)$),

$$G_u([k(t)]) = \left\langle \exp\left(\mathrm{i} \int_0^\infty \mathrm{d}t\, k(t) u(t) \right) \right\rangle. \tag{2.1}$$

Here $k(t)$ is a test function, and $\langle \cdots \rangle$ means an average over the stochastic realizations of $u(t)$ belonging to a given support. The knowledge of the characteristic functional allows a full characterization of the process. In fact, from this functional it is possible to calculate the whole Kolmogorov hierarchy, and of course any n-time moment and/or cumulant. This follows by defining the n-characteristic function $G_u^{(n)}\left(\{k_j, t_j\}_{j=1}^n\right)$ as

$$G_u^{(n)}\left(\{k_j, t_j\}_{j=1}^n\right) = G_u([k_\delta(t)]) \tag{2.2}$$

where the test function $k_\delta(t)$ must be taken as

$$k_\delta(t) = k_1 \delta(t - t_1) + \cdots + k_n \delta(t - t_n). \tag{2.3}$$

Thus, any n-time joint probability distribution $P\left(\{u_j, t_j\}_{j=1}^n\right) \equiv P_n(u_1, t_1; u_2, t_2; \cdots; u_n, t_n)$ can be obtained by Fourier inversion of $G_u^{(n)}\left(\{k_j, t_j\}_{j=1}^n\right)$

$$P\left(\{u_j, t_j\}_{j=1}^n\right) = \frac{1}{(2\pi)^n} \int \mathrm{d}k_1 \cdots \int \mathrm{d}k_n \exp\left(-\mathrm{i} \sum_{j=1}^n k_j u_j \right) G_u^{(n)}\left(\{k_j, t_j\}_{j=1}^n\right). \tag{2.4}$$

On the other hand, any n-time moment can be calculated as

$$\langle u(t_1) u(t_2) \cdots u(t_n) \rangle = (-\mathrm{i})^n \left. \frac{\partial^n G_u^{(n)}\left(\{u_j, t_j\}_{j=1}^n\right)}{\partial k_1 \partial k_2 \cdots \partial k_n} \right|_{k_j = 0}. \tag{2.5}$$

An equivalent formula holds for the cumulants (or correlation functions), in which case the differentiation must be taken from the logarithm of the n-characteristic function,

$$\langle\langle u(t_1) u(t_2) \cdots u(t_n) \rangle\rangle = (-\mathrm{i})^n \left. \frac{\partial^n \ln G_u^{(n)}\left(\{u_j, t_j\}_{j=1}^n\right)}{\partial k_1 \partial k_2 \cdots \partial k_n} \right|_{k_j = 0}. \tag{2.6}$$

We have shown [26, 27] that in order to obtain the characteristic functional of $u(t)$, it is only necessary to know the characteristic functional of the noise $\xi(t)$,

$$G_\xi([k(t)]) = \left\langle \exp\left(\mathrm{i} \int_0^\infty \mathrm{d}t\, k(t) \xi(t) \right) \right\rangle. \tag{2.7}$$

No other hypotheses were assumed over the noise. This general formalism allows us to deal with arbitrary non-Markovian evolution processes, where there is no clear underlying Fokker–Planck dynamics.

In what follows we will generalize our functional characterization for the case of a Langevin dynamics that includes memory effects, equation (1.1). We remark that no particular condition such as thermal equilibrium, Gaussian character, or any other, is imposed on the stochastic evolution of the processes $u(t)$.

2.2. The multivariable linear case with dissipative memory

A general case that covers many models of evolution, is the vectorial linear stochastic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(t) = -\int_0^t \mathrm{d}t'\boldsymbol{\Phi}(t-t')\cdot\mathbf{u}(t') + \boldsymbol{\xi}(t). \tag{2.8}$$

Here, $\mathbf{u}(t)$ represents a d-dimensional stochastic process $\{u_i(t)\}_{i=1}^d$, with $u_i \in (-\infty, +\infty)$. The memory kernel $\boldsymbol{\Phi}(t)$ is a $d\times d$ matrix with arbitrary kernel functions. The d-dimensional vector noise $\boldsymbol{\xi}(t)$ is characterized through its characteristic functional

$$G_{\boldsymbol{\xi}}([\mathbf{k}(t)]) = \left\langle \exp\left(\mathrm{i}\int_0^\infty \mathrm{d}t\,\mathbf{k}(t)\cdot\boldsymbol{\xi}(t)\right)\right\rangle \tag{2.9}$$

where the dot $\{\cdot\}$ denotes a scalar vector product, and $\mathbf{k}(t)$ is a vector of test functions $\{k_i(t)\}_{i=1}^d$.

We want to find an exact expression for the functional of the process $\mathbf{u}(t)$, defined as

$$G_{\mathbf{u}}([\mathbf{k}(t)]) = \left\langle \exp\left(\mathrm{i}\int_0^\infty \mathrm{d}t\,\mathbf{k}(t)\cdot\mathbf{u}(t)\right)\right\rangle. \tag{2.10}$$

The basic idea consists in writing the solution of equation (2.8) in terms of the corresponding Green function. Denoting the Laplace transform as $\tilde{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$, from equation (2.8) we get

$$s\tilde{\mathbf{u}}(s) - \mathbf{u}(0) = -\tilde{\boldsymbol{\Phi}}(s)\cdot\tilde{\mathbf{u}}(s) + \tilde{\boldsymbol{\xi}}(s). \tag{2.11}$$

This allows us to express the formal solution for each realization of the noises as

$$\mathbf{u}(t) = \langle\mathbf{u}(t)\rangle_0 + \int_0^t \mathrm{d}t'\boldsymbol{\Lambda}(t-t')\cdot\boldsymbol{\xi}(t') \tag{2.12}$$

where we have defined the vector

$$\langle\mathbf{u}(t)\rangle_0 \equiv \boldsymbol{\Lambda}(t)\cdot\mathbf{u}(0). \tag{2.13}$$

The $d\times d$ matrix $\boldsymbol{\Lambda}(t)$ is defined through its Laplace transform

$$\tilde{\boldsymbol{\Lambda}}(s) = \frac{1}{s\hat{\mathbf{I}} + \tilde{\boldsymbol{\Phi}}(s)} \tag{2.14}$$

where $\hat{\mathbf{I}}$ is a d-dimensional identity matrix and $\tilde{\boldsymbol{\Phi}}(s)$ is the Laplace transform of the matrix kernel $\boldsymbol{\Phi}(t)$. Equation (2.14) is equivalent to the evolution

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Lambda}(t) = -\int_0^t \mathrm{d}t'\boldsymbol{\Phi}(t-t')\cdot\boldsymbol{\Lambda}(t') \qquad \boldsymbol{\Lambda}(0) = 1. \tag{2.15}$$

From this equation it is possible to identify the matrix $\boldsymbol{\Lambda}(t)$ with the Green function of equation (2.8). Note that the vector $\langle\mathbf{u}(t)\rangle_0$ corresponds to the average value of the process $\mathbf{u}(t)$ for the case in which the average of the noise is zero, i.e., $\langle\boldsymbol{\xi}(t)\rangle = 0$.

After introducing the solution equation (2.12) into equation (2.10), and reordering the time integrals $\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' = \int_0^t \mathrm{d}t'' \int_0^t \mathrm{d}t'\Theta(t'-t'')$, where $\Theta(t)$ is the step function, we arrive at a closed expression for the characteristic functional of $\mathbf{u}(t)$ in terms of the functional of the vector noise. Thus, we get

$$G_{\mathbf{u}}([\mathbf{k}(t)]) = G_{\langle\mathbf{u}\rangle_0}([\mathbf{k}(t)])G_{\boldsymbol{\xi}}([\mathbf{z}(t)]) \tag{2.16}$$

where we have defined

$$G_{\langle\mathbf{u}\rangle_0}([\mathbf{k}(t)]) = \exp\left\{\mathrm{i}\int_0^\infty \mathrm{d}t\,\mathbf{k}(t)\cdot\langle\mathbf{u}(t)\rangle_0\right\} \tag{2.17}$$

and the vector test function $\mathbf{z}(t)$ as

$$\mathbf{z}(t) = \int_t^\infty \mathrm{d}t'\,\mathbf{k}(t') \cdot \mathbf{\Lambda}(t'-t). \tag{2.18}$$

Expression (2.16) gives us the desired exact functional $G_\mathbf{u}([\mathbf{k}(t)])$ as the product of two functionals. The first one corresponds to deterministic evolution, or equivalently to the averaged process when the noises have null averages. On the other hand, the second term comes from the noise characteristic functional, i.e., stochastic evolution.

Finally, the n-time characteristic function $G_\mathbf{u}^{(n)}\big(\{\mathbf{k}_j, t_j\}_{j=1}^n\big)$ can immediately be evaluated from the characteristic functional of the noise. The one-dimensional case, equation (2.2), is easily generalized to the vectorial case as

$$G_\mathbf{u}^{(n)}\big(\{\mathbf{k}_j, t_j\}_{j=1}^n\big) = G_\mathbf{u}([\mathbf{k}_\delta(t)]) \tag{2.19}$$

where the vectorial test function $\mathbf{k}_\delta(t)$ is

$$\mathbf{k}_\delta(t) = \mathbf{k}_1\delta(t-t_1) + \cdots + \mathbf{k}_n\delta(t-t_n). \tag{2.20}$$

Thus, using these last two equations and equations (2.16)–(2.18), we get

$$G_\mathbf{u}^{(n)}\big(\{\mathbf{k}_j, t_j\}_{j=1}^n\big) = \exp\left\{\mathrm{i}\sum_{j=1}^n \mathbf{k}_j \cdot \langle\mathbf{u}(t_j)\rangle_0\right\} G_{\boldsymbol{\xi}}([\mathbf{y}(t)]) \tag{2.21}$$

where the function $\mathbf{y}(t)$ reads

$$\mathbf{y}(t) = \sum_{j=1}^n \Theta(t_j - t)\mathbf{k}_j \cdot \mathbf{\Lambda}(t_j - t). \tag{2.22}$$

Equation (2.21) gives, in a simple way, the n-characteristic function of the process $\mathbf{u}(t)$ by evaluating the characteristic functional of the noise with the function $\mathbf{y}(t)$. This last function is defined in terms of the Green function $\mathbf{\Lambda}(t)$ of the problem. At this point, it is important to remark that our formalism is valid independent of the assumed form for the dissipative Green function evolution. Thus, this function may present a monotonous or oscillating decay (taking positive and negative values), and in general any one consistent with the corresponding kernel. In appendix A we show a particular two-dimensional stochastic process $\{u_i(t)\}_{i=1}^2$ where each component $u_i(t)$ represents the position and the velocity of a memory damped harmonic oscillator in the presence of an arbitrary noise structure.

3. Examples

Here we shall analyse, and characterize with our functional method, several amazing situations that arise by choosing different noise structures and a non-local dissipative kernel.

3.1. Local dissipation

From now on, we will be restricted to the one-dimensional case. By assuming a δ-Dirac correlated kernel

$$\Phi(t) = \gamma\delta(t) \tag{3.1}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = -\gamma u(t) + \xi(t). \tag{3.2}$$

Therefore, it is possible to obtain $\tilde{\Lambda}(s) = 1/(s+\gamma)$. Thus

$$\Lambda(t) = \exp[-\gamma t]. \tag{3.3}$$

With this solution, it is possible to recapture all the results obtained in [26].

3.2. Stable noises

Stable probability distributions play an important role in the theory of sums of random variables. This fact follows from the generalized central limit theorem valid for Lévy distributions and the asymptotic power-law decay [32, 33]. Here we apply our functional formalism to stable noises.

3.2.1. Gaussian noise. A zero-mean Gaussian noise $\xi(\tau)$, with an arbitrary correlation function $\sigma_\xi(\tau_2, \tau_1) = \langle \xi(\tau_2)\xi(\tau_1)\rangle$, is characterized by the functional [2]

$$G_\xi([k(t)]) = \exp\left(-\frac{1}{2}\int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \mathrm{d}\tau_1\, k(\tau_2)\sigma_\xi(\tau_2, \tau_1)k(\tau_1)\right). \tag{3.4}$$

Therefore, from equations (2.16)–(2.18) the characteristic functional of the process $u(t)$ results,

$$G_u([k(t)]) = G_{\langle u\rangle_0}([k(t)]) \exp\left(-\frac{1}{2}\int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \mathrm{d}\tau_1\, k(\tau_2)k(\tau_1)\sigma_u(\tau_2, \tau_1)\right) \tag{3.5}$$

where

$$\sigma_u(\tau_2, \tau_1) = \int_0^{\tau_2} \mathrm{d}\tau_b \int_0^{\tau_1} \mathrm{d}\tau_a\, \Lambda(\tau_2 - \tau_b)\sigma_\xi(\tau_b, \tau_a)\Lambda(\tau_1 - \tau_a) \tag{3.6}$$

is the correlation function of the Gaussian process $u(t)$. In fact, the n-time characteristic function of $u(t)$ is a multivariate Gaussian process.

3.2.2. Lévy noise. Another example that leads to conclusions similar to those using Gaussian noise, is Lévy noise. Its characteristic functional reads [21]

$$G_\xi([k(t)]) = \exp\left(-\frac{\sigma_\nu}{2}\int_0^\infty \mathrm{d}\tau |k(\tau)|^\nu\right) \tag{3.7}$$

where $0 < \nu < 2$. Thus, for example, the 1-time characteristic function of the process $u(t)$ reads

$$G_u^{(1)}(k, t) = \exp\{\mathrm{i}k\langle u(t)\rangle_0\} \exp\left(-\tfrac{1}{2}|k|^\nu\, \Xi_u(t)\right) \tag{3.8}$$

where

$$\Xi_u(t) = \sigma_\nu \int_0^t \mathrm{d}t' |\Lambda(t')|^\nu. \tag{3.9}$$

Therefore, memory-like linear Langevin equations driven by stable noises give rise to new stable stochastic processes whose correlation properties are defined in terms of the Green function $\Lambda(t)$ of the dissipative dynamics. Note that this result is valid independent of the particular form of the Green function.

3.3. Interplay between non-local friction and noise structure

In general, around a fixed point, the linearized dynamics of a dissipative system—depending on the parameters of the problem—may present a relaxation behaviour with quite different characteristics, such as, either a monotonous or a time-oscillating decay. Here we shall study the relaxation and the fluctuations of the process $u(t)$ by analysing the interplay between these dissipative characteristics and different noise structures.

The different forms of the dissipative dynamics can be modelled through a non-local exponential memory friction term

$$\Phi(t) = \delta \exp[-\lambda t] \tag{3.10}$$

therefore $\tilde{\Phi}(s) = \delta/(s+\lambda)$. From equation (2.14) we get $\tilde{\Lambda}(s) = (s+\lambda)/[s(s+\lambda)+\delta]$, which after inversion gives

$$\Lambda(t) = \exp(-\lambda t/2)\left\{\cos[w_0 t] + \frac{\lambda}{2w_0}\sin[w_0 t]\right\} \tag{3.11}$$

where the frequency w_0 is

$$w_0 = \sqrt{\delta - \left(\frac{\lambda}{2}\right)^2}. \tag{3.12}$$

As expected, the dissipative Green function, equation (3.11), has a decaying oscillatory regime. But there is also a possibility, for $\delta < \left(\frac{\lambda}{2}\right)^2$ for $\Lambda(t)$ to decay monotonously. Note that the local case (non-memory dissipation), equation (3.1), is re-obtained considering the limit $\lambda \to \infty$ and $\delta \to \infty$, with $\delta/\lambda \to \gamma$ (finite).

Now we present some results using the Green function equation (3.11) in connection with two different structures for the fluctuating term, i.e., a radioactive noise and a Poisson white noise. We will show that the interplay between the deterministic Green function and the structure of the noise plays a crucial role to determine the relaxation and, in general, any statistical property of the stochastic process $u(t)$. Later on, a memory kernel with a power-law decay will be studied in the context of fractional derivatives in section 3.5.

3.3.1. Radioactive decay noise. The radioactive noise is non-white and Markovian. Its 1-time and conditional probability evolve, controlled by the master equation

$$\frac{\mathrm{d}P_\xi(t)}{\mathrm{d}t} = \beta[(\xi+1)P_{\xi+1}(t) - \xi P_\xi(t)]. \tag{3.13}$$

Thus, the realization of this noise starts with some initial value ξ_0 and at random times decreases by finite unit steps until the process reaches the zero value. The constant β defines the probability per unit time for such discrete steps (β-decay). The characteristic functional reads [2]

$$G_\xi([k(t)]) = \left(\beta\int_0^\infty \mathrm{d}\tau \exp\left[-\beta\tau + \mathrm{i}\int_0^\tau \mathrm{d}\tau' k(\tau')\right]\right)^{\xi_0}. \tag{3.14}$$

This noise clearly does not reach a stationary regime. Using this noise, the 1-time characteristic function of the process $u(t)$ is

$$G_u^{(1)}(k,t) = \exp(\mathrm{i}k\langle u(t)\rangle_0)\left(\mathrm{e}^{-\beta t}\exp\left[\mathrm{i}k\int_0^t \mathrm{d}\tau'\Lambda(\tau')\right]\right.$$
$$\left. + \beta\int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\beta\tau}\exp\left[\mathrm{i}k\int_{t-\tau}^t \mathrm{d}\tau'\Lambda(\tau')\right]\right)^{\xi_0}. \tag{3.15}$$

This noise leads to very striking dynamical behaviour for the driven process $u(t)$. In figures 1 and 2 we have plotted, at different times, the corresponding exact probability distribution $P(u,t)$, which we found anti-transforming equation (3.15) by using a fast Fourier algorithm. In figure 1, the parameters for the memory kernel were chosen as $\delta = 0.1$, $\lambda = 1$. Thus, the Green function decays without oscillations (see inset). The noise decay rate is $\beta = 1$ and its initial value was chosen to be $\xi_0 = 2$. At time $t = 1$ (full line), we see that the probability distribution is highly irregular and also has a δ-Dirac contribution (indicated symbolically with the vertical line). This singular behaviour arises from the particular realization $\xi(t)$ in which the noise has not yet suffered any decay.

Due to the non-vanishing value of the noise average, $\langle\xi(t)\rangle \neq 0$, the δ-Dirac contribution has different locations at different times. This term can be directly read from the characteristic

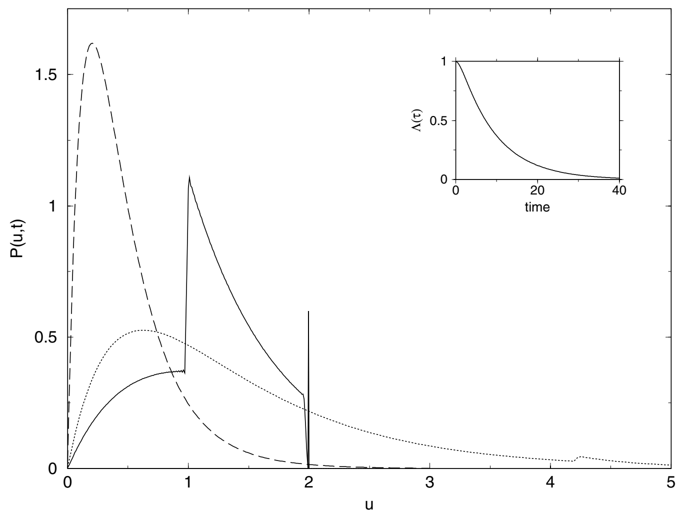

Figure 1. Probability distribution, $P(u, t)$, for a memory-like driven process $u(t)$ with an exponential dissipative kernel and in the presence of a radioactive noise, $\xi(t)$, as a function of u for three different times, $t = 1$ (full line), $t = 5$ (dotted line), $t = 15$ (dashed line), taken in arbitrary units. The initial condition was chosen as $u(t = 0) = 0$. The noise parameters are $\beta = 1$ and $\xi_0 = 2$. The parameters of the Green function are $\delta = 0.1$ and $\lambda = 1$; the inset shows its monotonic decaying behaviour as a function of time. The straight line in $P(u, t = 1)$ indicates the δ-Dirac contribution (at short times) in the distribution of the process $u(t)$. Asymptotically, $P(u, t \to \infty)$ goes to a δ-Dirac located at $u = 0$.

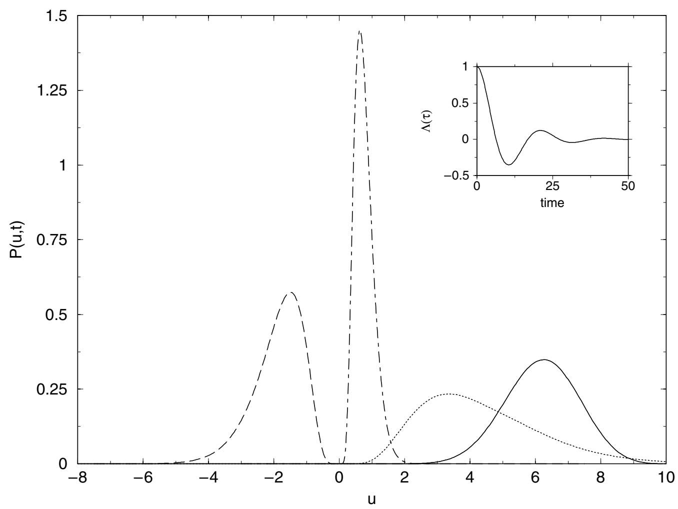

Figure 2. Probability distribution $P(u, t)$ as in figure 1 for four different times, $t = 1$ (continuous line), $t = 5$ (dotted line), $t = 15$ (dashed line), $t = 25$ (dot-dashed line), in arbitrary units. The noise parameters are $\beta = 1$ and $\xi_0 = 10$. The parameters of the Green function are $\delta = 0.1$ and $\lambda = 0.2$; the inset shows its decaying oscillatory behaviour as a function of time. Here, at large times the distribution $P(u, t \to \infty)$ also goes asymptotically to a δ-Dirac located at $u = 0$.

functional of the process $u(t)$, equation (3.15), which gives $\exp(-\xi_0\beta t)\,\delta\left[u - \xi_0 \int_0^t \mathrm{d}\tau' \Lambda(\tau')\right]$. The discontinuity of the probability distribution $P(u, t)$ is due to the strong change that arises in the driving noise after a β-decay. In fact, in this example, the noise intensity decreases to half of its initial value. At a later time, $t = 5$ (dotted line), due to the dissipative dynamics the probability distribution $P(u, t)$ loses it discontinuous character and seems to be accumulated near the origin. At even later times, $t = 15$ (dashed line) this accumulation seems to increase. In fact, the stationary distribution is a δ-Dirac centred in $u = 0$. This behaviour is a consequence of the coupling between the noise properties and the deterministic dynamics of $u(t)$: as the noise intensity vanishes at long times, all realizations $u(t)$ are attracted by the stable point of the dissipative dynamics, which corresponds to $u = 0$.

In figure 2 we show the case in which the Green function is oscillatory in time (see inset). The parameters are $\delta = 0.1, \lambda = 0.2$, and for the noise we choose $\beta = 1$ and $\xi_0 = 10$. Here, in contrast to figure 1, we have increased the initial intensity of the noise, which implies that the δ-Dirac contribution decays very fast. On the other hand, this higher initial value implies that the 'discrete' decay of the corresponding noise realizations does not change appreciably the noise intensity. In consequence, the distribution $P(u, t)$ will be smooth for all times. As in the previous example, all realizations of the process $u(t)$ are attracted by the stable point $u = 0$. Nevertheless, here the transient behaviour reflects the oscillatory behaviour of the Green function. In fact, we note that at successive times, the *centre of mass* of the distribution $P(u, t)$ follows approximately the dissipative dynamics of the Green function oscillating around the origin $u = 0$ (see times: $t = 1$ continuous line, $t = 5$ dotted line, $t = 15$ dashed line, $t = 25$ dot-dashed line). In addition, we note that the width of the distribution grows approximately up to $t = 5$. In fact, at this time most of the noise realizations have decayed to a null value. For later times, the distribution is mainly governed by the dissipative dynamics of $u(t)$. Therefore, its width decreases up to reaching a δ-Dirac form in the stationary state.

We remark that for this noise the stationary distribution $P(u)$ is always determined by the fixed point of the dissipative dynamics. This characteristic is due to the vanishing noise amplitude in the long-time regime. In the next example, we show a case where the noise properties participate, in a crucial way, in both the transient and the stationary properties of the process $u(t)$.

3.3.2. Shot noise. The stochastic realizations of shot noise are defined by a sequence of pulses $\psi(t)$, each one arriving at random independent times t_i. The characteristic functional of the noise [2, 26, 27] $\xi(t)$, in the interval $t \in [0, \infty]$, is

$$G_\xi([k(t)]) = \exp\left(\int_0^\infty d\tau\, q(\tau)\left[\exp\left(\mathrm{i}\int_0^\infty k(t)\psi(t-\tau)\,dt\right) - 1\right]\right) \qquad (3.16)$$

where $q(\tau)$ is the density of arriving pulses. Therefore, the complete characterization of the process $u(t)$ is given by

$$G_u([k(t)]) = G_{\langle u\rangle_0}([k(t)])\exp\left(\int_0^\infty d\tau\, q(\tau)\left[\exp\left(\mathrm{i}\int_0^\infty k(t)\Psi(t,\tau)\,dt\right) - 1\right]\right) \qquad (3.17)$$

where the function $\Psi(t, \tau)$ is defined as

$$\Psi(t,\tau) = \int_0^t dt'\,\Lambda(t-t')\psi(t'-\tau). \qquad (3.18)$$

If $\psi(t) = A\delta(t)$ and the density of 'dots' $q(\tau)$ is uniform $q(\tau) = \rho$ the noise $\xi(t)$ reduces to the more familiar white shot-noise or Poisson noise whose characteristic functional is

$$G_\xi([k(t)]) = \exp\left(\rho\int_0^\infty dt[\exp[\mathrm{i}Ak(t)] - 1]\right). \qquad (3.19)$$

By adding two statistically independent white shot-noises, with opposite amplitudes, we get

$$G_\xi([k(t)]) = \exp\left(2\rho\int_0^\infty \mathrm{d}t[\cos(Ak(t)) - 1]\right). \tag{3.20}$$

This functional characterizes a noise with a zero average, whose realization consists of the random arrival of δ-Dirac pulses with amplitude $\pm A$. Note that in the limit $A \to 0$, $\rho \to \infty$ with $A^2\rho = D/2$, this symmetrical white shot-noise converges to a Gaussian white noise, i.e., equation (3.4) with $\sigma_\xi(\tau_2, \tau_1) = D\delta(\tau_2 - \tau_1)$.

Using equation (3.20), the 1-time characteristic function of the process $u(t)$ is given by

$$G_u^{(1)}(k, t) = \exp\{\mathrm{i}k\langle u(t)\rangle_0\}\exp\left(2\rho\int_0^t \mathrm{d}\tau[\cos(Ak\Lambda(\tau)) - 1]\right). \tag{3.21}$$

As we will show below, this expression leads to a rich variety of possible stochastic dynamical behaviour for the process $u(t)$; both, in the transient as in the stationary state, the different behaviours arise from the competition between the different characteristic time scales of the Green function (equation (3.11)) and those of the noise. In the monotonic regime, $\delta < \lambda^2/4$, the decay of $\Lambda(\tau)$ can be characterized by the rate δ/λ. For $\delta > \lambda^2/4$, when $w_0 > \lambda/2$, the oscillatory behaviour of $\Lambda(\tau)$ is characterized by the damping rate $\lambda/2$ and the frequency w_0. On the other hand, the noise is characterized by the rate ρ and the amplitude A. The analysis of the dependence of $P(u, t)$ on these parameters can be simplified by defining the rescaled process $u'(t) = u(t)/A$, whose evolution can be written as

$$\frac{\mathrm{d}u'(\tau')}{\mathrm{d}\tau'} = -\delta'\int_0^{\tau'} \mathrm{d}\tau'' \exp[-(\tau' - \tau'')]u'(\tau'') + \xi'(\tau'). \tag{3.22}$$

Here, we have defined the dimensionless time $\tau' = \lambda t$ and parameter $\delta' = \delta/\lambda^2$. Furthermore, the dimensionless symmetric Poisson noise $\xi'(\tau')$ is characterized only by the dimensionless ratio $\rho' = \rho/\lambda$. Thus, the full dynamical properties of the process $u(t)$ are controlled by the parameters δ' and ρ'.

In figure 3 we have obtained numerically $P(u, t)$ from fast Fourier transform of equation (3.21). In figure 3 the parameters of the kernel are $\delta = 0.2, \lambda = 1$, and for the noise we used $A = 1$, $\rho = 7.5$. For these values, the frequency w_0 in equation (3.12) is complex. Therefore, the Green function $\Lambda(t)$ decays monotonically in time (see inset). At short times, $t = 0.5$ (figure 3(*a*)), the probability distribution presents a series of peaks, separated by a distance $A = 1$. This property is a direct consequence of the nature of shot-noise, where each arriving δ-Dirac pulse produces a shift of magnitude A in the driven process $u(t)$. In this short-time regime the effect of the dissipative dynamics is negligible. At later times, the dissipative contribution becomes appreciable; its effect is to attract all the realizations of $u(t)$ to the fixed point $u = 0$. This action, added to the shift effect of the arriving pulses, produces a widening of all the peaks of the distribution; then any value of u becomes (approximately) probable. This effect is clearly seen in figure 3(*b*) ($t = 1.65$). Note that the distribution retains its peaky structure. This erasing effect is increased over time; see figure 3(*c*) (time $t = 2$). At later times, all the peaks have disappeared and the distribution goes asymptotically to a Gaussian distribution; see figure 3(*d*) (times $t = 2.5$ (dotted line) and $t = 20$ (full line)).

In general, an asymptotic Gaussian stationary distribution is always expected when the rate of the arriving δ-Dirac pulses is greater than the characteristic decay rate of the Green function. When the decay of $\Lambda(t)$ is monotonic, this condition can be expressed through the inequality $\rho' > \delta'$, which is clearly satisfied in the previous figure 2. When this relation is not satisfied, deviations from a Gaussian distribution will arise both in the transient as in the stationary distribution. We show this effect in figure 4.

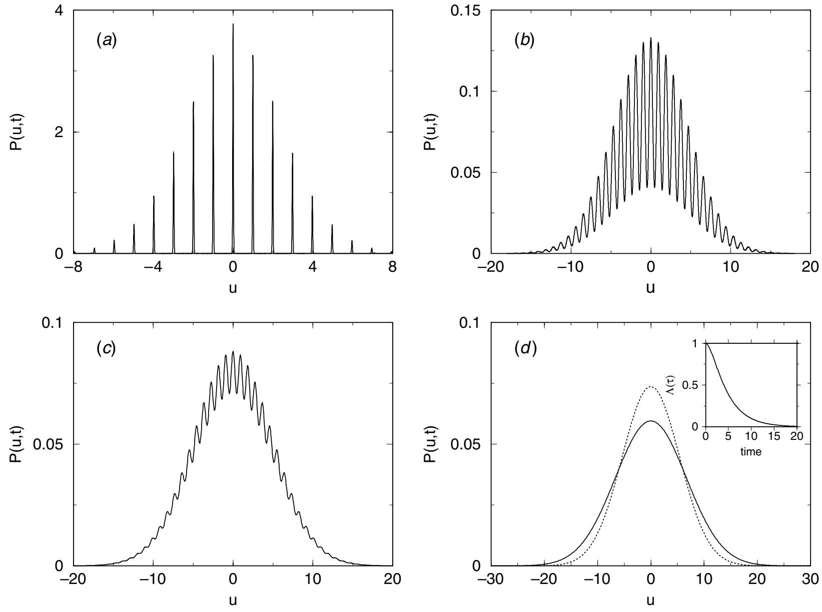

Figure 3. Probability distribution, $P(u, t)$, for a memory-like driven process $u(t)$ with an exponential dissipative kernel and in the presence of a symmetric white Poisson noise, $\xi(t)$, as a function of u for five different times, (a) $t = 0.5$, (b) $t = 1.65$, (c) $t = 2$, (d) $t = 2.5$ (dotted line) and $t = 20$ (full line), taken in arbitrary units. The initial condition was chosen as $u(t = 0) = 0$. The noise parameters are $A = 1$, $\rho = 7.5$. The parameters of the Green function are $\delta = 0.2$, $\lambda = 1$; the inset in (d) shows its monotonic decaying behaviour as a function of time.

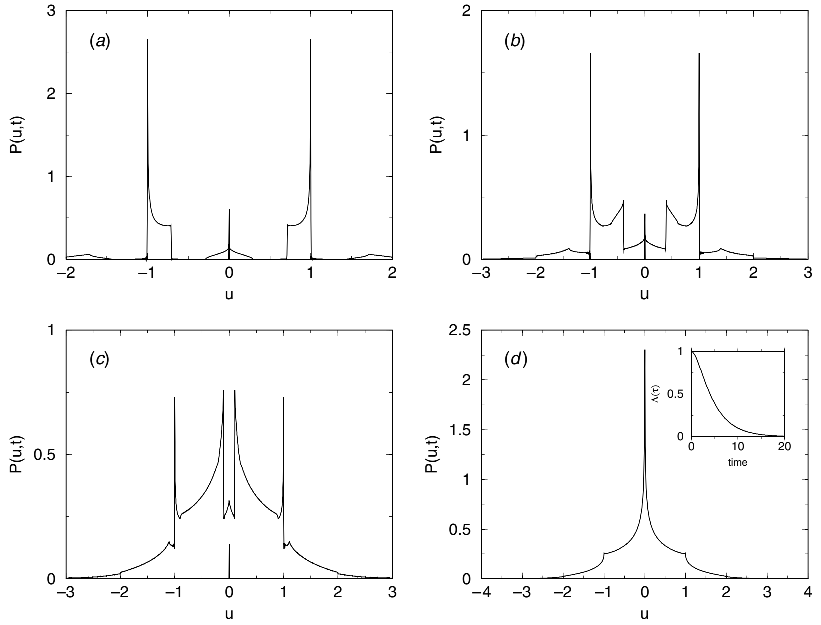

Figure 4. Probability distribution $P(u, t)$ as in figure 3, for four different times, (a) $t = 2.5$, (b) $t = 5$, (c) $t = 10$ and (d) $t = 100$ where the stationary state is practically reached. The noise parameters are $A = 1$ and $\rho = 0.1$. The parameters of the Green function are as in figure 3; the inset in (d) shows its monotonic decaying behaviour as a function of time. The δ-Dirac contributions (see text) are indicated with straight lines.

In figure 4, the parameters of the Green function (see inset) are the same as in figure 3, ($\delta = 0.2$, $\lambda = 1$), and for the noise we have used $A = 1$, $\rho = 0.1$. As in the previous case, the behaviour of $P(u, t)$ arises from the competition between the shift effect of the arriving pulses and the dissipative dynamics of $u(t)$. For example, for short times (figure 4(*a*), $t = 2.5$) the probability shows a set of maxima located at $\pm A$ with tails pointing towards the origin. These tails are mainly originated by the dissipative dynamics that attracts all realizations to the stable point $u = 0$. On the other hand, here the occupation around $u = 0$ comes from the shift effect produced by two arriving pulses with opposite sign. In contrast to the previous example, at later times (figure 4(*b*), $t = 5$) the evolution of the distribution $P(u, t)$ is mainly governed by the dissipative dynamics and the structure of peaks is retained throughout the evolution, figure 4(*c*) $t = 10$, and figure 4(*d*) $t = 100$, where the stationary state is practically reached. Note that in this stationary state, the non-Gaussian characteristics are present in $u \approx 0$ and in $u \approx \pm A$.

We note that during all the transient dynamics, there exists a δ-Dirac contribution (indicated with the vertical lines) centred at the origin, $u = 0$. This term is originated by the noise realization in which no pulse has arrived up to time t. Therefore, it is exponentially damped with a rate 2ρ, i.e., it can be written as $\exp(-2\rho t)\delta(u)$. At each of the chosen times, the weights of the δ-Dirac term are, respectively, 0.606, 0.368, 0.135 and 2×10^{-9}. Note that in the stationary state, the δ-Dirac contribution is always completely washed out. We remark that, independent of the values of the parameters, this δ-Dirac contribution is always present. Nevertheless, note that in the case of figure 3, it is rapidly attenuated and its contribution is insignificant in the time scale of that plot.

In figures 3 and 4 we have assumed a monotone decreasing Green function. In general, the interplay between an asymmetric Poisson noise, equation (3.19), and an oscillatory Green function is similar to that found in the case of radioactive noise. In the case of a symmetric noise, equation (3.21), due to the symmetry of the problem, the transient dynamics is approximately similar to that found for a monotone decay. Thus, the more relevant aspect to analyse is the corresponding stationary distribution.

In figure 5 we study the properties of the stationary distribution $P(u)$ for different values of the parameter of the Green function and the noise. In figure 5(*a*) the parameters corresponding to the Green function (see inset) are $\delta = 2$, $\lambda = 1$, while for the Poisson noise we have taken $\rho = 0.3$ (dotted line), $\rho = 0.4$ (dashed line) and $\rho = 0.8$ (continuous line); in all cases using $A = \sqrt{0.4/\rho}$. In consequence, the effective amplitude of the noise ρA^2 is constant in the three graphs. Due to this selection the tails of the distributions coincide, indicating that all the distributions are asymptotically, for large $|u|$, Gaussian. This property will be analysed analytically in the next subsection.

As in the case of the non-oscillating Green function, figures 3 and 4, here we expect a Gaussian stationary distribution when the average waiting time between two arriving pulses is shorter than the characteristic relaxation time of the Green function. In this case, the oscillatory decay of the Green function is characterized by the rate $\lambda/2$. Thus, we estimate that for $\rho' > 1/2$ a Gaussian statistics arises. This prediction agrees very well with the plots shown in figure 5(*a*). Nevertheless, note that this inequality does not depend on the frequency of the Green function oscillations. In figure 5(*b*), we will check the validity of this prediction.

In figure 5(*b*) we show the stationary distribution $P(u)$ when keeping the noise parameters fixed and changing the values of the parameters of the Green function. The noise parameters were chosen as $\rho = 0.25$, $A = 1$ and for the Green function $\lambda = 1$, $\delta = 0.15$ (full line), $\delta = 0.6$ (dashed line) and $\delta = 50$ (dotted line). In the inset, we show the corresponding Green function for each value. Consistently with our previous analysis, we confirm that independent of the value of the characteristic frequency of the Green function, the stationary distribution

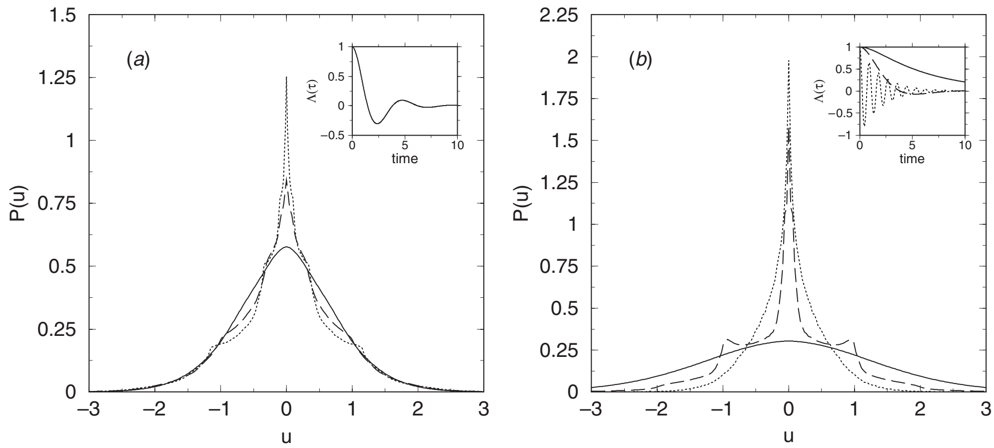

Figure 5. Stationary probability distribution, $P(u)$, for a memory-like driven process $u(t)$ with an exponential dissipative kernel and in the presence of symmetric white Poisson noise, as a function of u for different values of the parameters for the Green function and the noise. In (*a*) $\delta = 2$, $\lambda = 1$, while for the Poisson noise we take $\rho = 0.3$ (dotted line), $\rho = 0.4$ (dashed line) and $\rho = 0.8$ (continuous line) and $A = \sqrt{0.4/\rho}$. In (*b*) the noise parameters are the same in each plot, $\rho = 0.25$, $A = 1$, but we change the parameters for the Green function, $\lambda = 1$, $\delta = 0.15$ (full line), $\delta = 0.6$ (dashed line) and $\delta = 50$ (dotted line). The insets show the Green functions for the corresponding different parameters.

$P(u)$ is non-Gaussian for $\rho' < 1/2$. Furthermore, we note that by increasing the frequency of the Green function, the fast oscillations of $\Lambda(t)$ completely wash out the non-Gaussian peaks located at $\pm A$, only one peak remaining centred at $u = 0$. On the other hand, with the chosen values of the noise parameters, a Gaussian distribution only arises for a non-oscillating Green function.

Finally, we want to remark that the criteria for obtaining a stationary Gaussian distribution $\rho' > \delta'$, valid when the decay of the Green function is monotone, and $\rho' > 1/2$ valid for an oscillatory decay, change their validity gradually around the value $\delta' = 1/4$, which corresponds to the point where the Green function modifies its characteristic decay.

3.4. *Long-tail stationary distributions*

Here we are interested in characterizing the stationary distribution of the stochastic process $u(t)$; it is of great importance to know whether there will appear or not a long-tail in the distribution. In order to make a general analysis, here we assume that the structure of the noise is such that we can write the 1-time characteristic function in the form

$$G_u^{(1)}(k_1, t_1) = \exp\left\{\int_0^{t_1} \mathrm{d}t\ f(k_1\Lambda(t))\right\} \tag{3.23}$$

where $f(z)$ is an arbitrary function. Note that this structure is compatible only with white noise. In fact, equation (3.23) follows using the test function $z(t) = \Theta(t_1 - t)\Lambda(t_1 - t)k_1$ in the general expression for the functional of $u(t)$, equation (2.16), and assuming $G_\xi[k(t)] = \exp\left(\int_0^\infty \mathrm{d}t f(k(t))\right)$. Taking the logarithm in equation (3.23), and introducing the Laplace transform (here denoted as $\mathcal{L}_s$) we get the general expression

$$\ln G_u^{(1)}(k, t = \infty) = \lim_{s\to 0} s\mathcal{L}_s\left[\ln G_u^{(1)}(k, t)\right] = \lim_{s\to 0} \mathcal{L}_s[f(k\Lambda(t))]. \tag{3.24}$$

If a Taylor expansion of $f(z)$ exists, and we can commute the Taylor expansion and the Laplace operator, we arrive at

$$\ln G_u^{st}(k) = \sum_{n=1}^{\infty} C_n k^n \mathcal{L}_s[\Lambda(t)^n]_{s=0}. \tag{3.25}$$

This formula characterizes the stationary distribution of the stochastic process $u(t)$, if it exists, as a series expansion in the Fourier component k.

From equation (3.25) it is simple to see that even when the stochastic transient can be non-Gaussian (depending on the driving noise) in the large asymptotic scaling $| u |\to \infty$, the Fourier transform of the stationary distribution goes as

$$G_u^{st}(k) \sim \exp(\mathrm{i}kA_0 - k^2 B_0 + \cdots) \qquad \text{for } k \sim 0 \tag{3.26}$$

where A_0, B_0 are constants given in terms of the dissipative memory and the structure of the noises; then over a large scale the behaviour is not anomalous. The analysis with a non-white noise is similar and the general conclusion is the same. So in order to understand the occurrence of anomalies or long tails in the stationary distribution of the process $u(t)$ we have to consider driving noises that do not fulfil the hypothesis (3.25). This case is achieved by structures like that from the Lévy noise, see equation (3.7), or its associated one-side power-law noise shown in appendix B.

In the previous subsection we have shown that Poisson noise, equation (3.19), can give rise, during the transient and in the stationary state, to strongly non-Gaussian distributions for the process $u(t)$. Nevertheless, this noise does not give rise to long-tail stationary distributions. As a matter of fact, the characterization of the stationary state induced by this shot-noise follows straightforwardly using equation (3.24). From equation (3.19) it follows that

$$\ln G_u^{st}(k) = \lim_{t\to\infty} \rho \int_0^t \mathrm{d}\tau (\exp[\mathrm{i}Ak\Lambda(\tau)] - 1)$$
$$= \frac{\rho}{\gamma}\{\mathrm{Ei}(\mathrm{i}Ak) - \ln(A|k|) - \mathcal{E}\}$$

where $\mathrm{Ei}(x)$ is the exponential integral function [34], and $\mathcal{E}$ is the Euler constant. For simplification, we have assumed the local Green function (3.3). Here it is interesting to note the non-analytic structure of $G_u^{st}(k)$, perhaps pre-announcing an anomalous behaviour in the stationary distribution of the process $u(t)$. Nevertheless, it should be noted that in the large scale limit $k \to 0$ this non-analytic structure cancels out, leading therefore to an expansion like (3.26), which implies the absence of any long-tail distributions. This is not what happens if the driving noise has a long-tail structure such as that we discuss below.

3.4.1. Abel noise. In close connection with a Poisson noise with a random density of arriving pulses, in appendix B we have defined the characteristic functional of Abel noise $\xi(t)$. The functional of this noise reads ($t \in (0, \infty)$)

$$G_\xi([k(t)]) = \frac{2}{\Gamma(\mu)}\left(\sqrt{a\int_0^\infty (1-\mathrm{e}^{\mathrm{i}k(t)})\,\mathrm{d}t}\right)^{\mu} K_\mu\left(2\sqrt{a\int_0^\infty (1-\mathrm{e}^{\mathrm{i}k(t)})\,\mathrm{d}t}\right) \tag{3.27}$$

where $K_\mu(z)$ is the Basset function[3] and $\Gamma(\mu)$ the gamma function, see [34]. Note that in contrast with Lévy noise, this functional has well-defined integer moments $\langle\xi(t)^q\rangle$ if $\mu > q$.

[3] Alternative names for the Basset functions are the modified Bessel functions of the third kind, Bessel's functions of the second kind of imaginary argument, Macdonald's functions and the modified Hankel functions.

Using this noise, the 1-time characteristic function of the process $u(t)$ reads

$$G_u^{(1)}(k,t) = \frac{2\exp(\mathrm{i}k\langle u(t)\rangle_0)}{\Gamma(\mu)}\left(\sqrt{a\int_0^t (1-\exp(\mathrm{i}k\Lambda(\tau)))\,\mathrm{d}\tau}\right)^{\mu} \times K_\mu\left(2\sqrt{a\int_0^t (1-\exp(\mathrm{i}k\Lambda(\tau)))\,\mathrm{d}\tau}\right). \tag{3.28}$$

From this expression (if $\Lambda(t) > 0, \forall t$) it is possible to see that the asymptotic behaviour of the probability distribution is characterized by a one-sided power-law distribution of the Abel form

$$P(u,t) \sim \frac{\mathcal{A}(t)^\mu}{\Gamma(\mu)}\frac{\exp(-\mathcal{A}(t)/u)}{u^{1+\mu}} \quad \text{for large positive } u \tag{3.29}$$

where

$$\mathcal{A}(t) = a\int_0^t \Lambda(\tau)\,\mathrm{d}\tau > 0. \tag{3.30}$$

In the case of local dissipation, using the Green function (3.3), we get

$$\mathcal{A}(t) = \frac{a}{\gamma}(1-\exp(-\gamma t)). \tag{3.31}$$

From this result it is simple to see that the (one-sided stable) directed random walk behaves like $P(u,t) \sim (at)^\mu/u^{1+\mu}$, as was expected; see appendix B.

In the presence of non-local dissipation and in the case $\delta > \left(\frac{\lambda}{2}\right)^2$, due to the fact that the Green function (3.11) changes its sign oscillating in time, a power-law behaviour is obtained in a quite unusual way during the transient, i.e., the long-tail changes its support from a positive to a negative domain and so on as the time goes on. On the other hand, in the stationary regime we get

$$\mathcal{A}(t=\infty) = \frac{a\lambda}{\delta} \tag{3.32}$$

and then also a power law is reached at long times, with its long tail in the positive domain, i.e., for $u > 0$, $P(u, t \to \infty) \sim \left(\frac{a\lambda}{\delta}\right)^\mu u^{-1-\mu}$.

In figure 6 we show this amazing behaviour for the noise parameters $a = 3, \mu = \frac{1}{2}$, $K_{1/2}(x) = \sqrt{\pi/2x}\exp(-x)$. In this case we have used for the Green function (see inset) the parameter values $\delta = 0.85$ and $\lambda = 0.2$. At time $t = 3.5$ (figure 6(*a*)) we can see that there is a long tail in the positive domain, but also a sort of exponential decay appears in the negative domain as a result of the interplay between the oscillating Green function and the Abel noise. At later time $t = 5$ (figure 6(*b*)) it is possible to see that the long tail has changed its support, showing also an exponential decay but now in the positive domain. At time $t = 50$ (figure 6(*c*)) the stationary distribution is almost reached with the expected long tail in the positive domain. In these figures we have also fitted (with dashed lines) the corresponding long tails with the asymptotic behaviour

$$P(u,t) \sim \left|\frac{\Gamma(-\mu)}{\Gamma(\mu)}\right| \frac{|\mathcal{A}(t)|^\mu}{[\mathrm{sig}(u)\cdot u]^{1+\mu}} \tag{3.33}$$

where $\mathrm{sig}(u)$ represents the sign of u.

To end this subsection let us call the attention that by subtracting two statistically independent Abel noises, it is also possible to obtain the n-time characteristic function of a symmetric process $u(t)$, see appendix B.

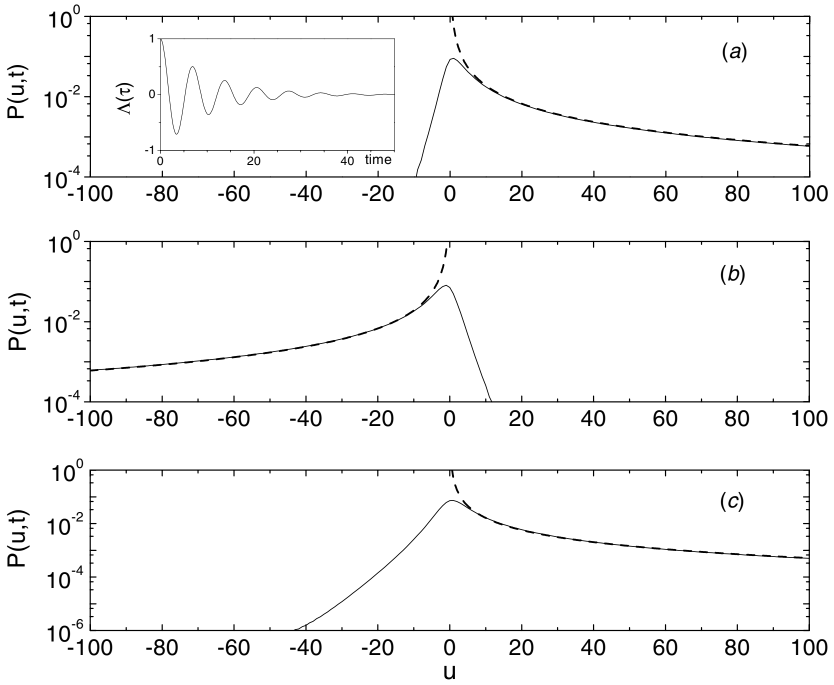

Figure 6. Probability distribution, $P(u, t)$, for a memory-like driven process $u(t)$ with an exponential dissipative kernel and in the presence of the Abel noise, $\xi(t)$, as a function of u for three different times in arbitrary units, (*a*) $t = 3.5$, (*b*) $t = 5$, (*c*) $t = 50$. The initial condition was $u(t = 0) = 0$. The noise parameters are $a = 3$ and $\mu = \frac{1}{2}$. The parameters of the Green function are $\delta = 0.85$ and $\lambda = 0.2$; the inset shows its decaying oscillatory behaviour as a function of time. The dashed lines correspond to the long-tail fit (see text). In (*c*) the stationary regime of the distribution of the process $u(t)$ has been reached.

3.5. Fractional derivative evolutions

Another example that can be characterized with the present functional technique is the case of fractional Langevin equations with dissipation. There exist many different ways of introducing this kind of equations. Here we will consider the evolution

$$ {}_0D_t^{\alpha}[u(t)] - u_0\frac{t^{-\alpha}}{\Gamma(1-\alpha)} = -\eta^{\alpha}u(t) + \xi(t) \tag{3.34} $$

with $1 \geqslant \alpha > 0$. This evolution was proposed to simulate the dynamics of financial markets where it was found that non-Gaussian driving noises should be the suitable ones [23, 24]. The proper interpretation of this fractional stochastic equation is actually an integral equation

$$ u(t) - u_0 = -\eta^{\alpha}{}_0D_t^{-\alpha}[u(t)] + {}_0D_t^{-\alpha}[\xi(t)]. \tag{3.35} $$

Here, u_0 is the initial condition, and ${}_0D_t^{-\alpha}$ is the Riemann–Liouville integral operator [35]

$$ {}_0D_t^{-\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)}\int_0^t \mathrm{d}\tau \frac{f(\tau)}{(t-\tau)^{1-\alpha}}. \tag{3.36} $$

It has been proved [23] that the solution of this equation for each realization of the noise can be written as

$$ u(t) = \langle u(t)\rangle_0 + \int_0^t \mathrm{d}t'\Lambda(t-t')\xi(t') \tag{3.37} $$

where

$$\langle u(t)\rangle_0 = u_0 E_{\alpha,1}[-(\eta t)^\alpha]. \tag{3.38}$$

On the other hand, the Green function $\Lambda(t)$ is given by

$$\Lambda(t) = \Theta(t) t^{\alpha-1} E_{\alpha,\alpha}[-(\eta t)^\alpha]. \tag{3.39}$$

Note that for this model of dissipative fractional equations the average $\langle u(t)\rangle_0$ and the noise propagate with different Green functions. The generalized Mittag–Leffler function $E_{\alpha,\beta}$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \qquad \alpha > 0, \beta > 0. \tag{3.40}$$

From equation (3.37) it is evident that our functional approach allows us to introduce any kind of statistics to drive the fractional stochastic evolution.

3.5.1. Competition between fractional derivative and Lévy noise. Here we are interested in analysing the competition between the statistics of Lévy noise and the dynamical effects introduced by a fractional derivative structure. In order to get a simpler analysis, here we will assume zero dissipation, i.e., $\eta = 0$. In this case, the previous evolution reduces to

$$\langle u(t)\rangle_0 = u_0 \qquad \Lambda(t) = \frac{\Theta(t)}{\Gamma(\alpha)} \frac{1}{t^{1-\alpha}}. \tag{3.41}$$

Note that upon differentiation of equation (3.35), these two previous expressions characterize the evolution

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = {}_0D_t^{1-\alpha} \xi(t). \tag{3.42}$$

We would like to stress that this problem in the frame of our functional approach can be done in a simple way. Using equation (3.41) and the functional of a Lévy white noise, equation (3.7), the process $u(t)$ is fully characterized by the function (see equation (3.9))

$$\Xi_u(t) = \sigma_\nu \left(\frac{1}{\Gamma(\alpha)}\right)^\nu \left(\frac{1}{1-\nu(1-\alpha)}\right) t^{1-\nu(1-\alpha)}. \tag{3.43}$$

This expression is only well defined if $\nu(1-\alpha) < 1$. Note the restriction on the parameters $\{\nu, \alpha\}$. This result contrasts with that of a driving Gaussian white noise, where the restriction is $\alpha \in (\frac{1}{2}, 1]$. Thus, in the case of Lévy noise, smaller values of the parameter $\alpha \in (0, 1]$ are allowed at the expense of diminishing the value of $\nu \in (0, 2]$ (i.e., large Lévy-step excursions).

Finally, we remark that equation (3.42) driven by Gaussian fluctuations, can be mapped with an effective medium approximation in the context of disordered systems [3][4]. Note that from our functional approach, it is easy to calculate correlation functions and more complicated objects than the propagator of the system; this is something that is hard to obtain in the context of self-consistent approximations [36].

[4] This mapping can be done through the corresponding generalized diffusion coefficient $D_\alpha(s) = (s^2/2)\mathcal{L}_s[\langle\langle u(t)u(t)\rangle\rangle]$, which in the present case reads

$$D_\alpha(s) = \left(\frac{\kappa_\alpha}{\Gamma(\alpha)}\right)^2 \left(\frac{\Gamma(2\alpha)}{2\alpha-1}\right) \frac{s^{2(1-\alpha)}}{2} \qquad \alpha \in \left(\frac{1}{2}, 1\right].$$

On the other hand, from an effective medium approximation [36], the self-consistent Laplace-dependent diffusion coefficient in $1D$ is given, asymptotically, by

$$D(s) \approx \left(\frac{\sin \pi\zeta}{2\pi(1-\zeta)}\right)^{2/(2-\zeta)} s^{\zeta/(2-\zeta)}.$$

4. Summary and conclusions

We have completely characterized generalized linear Langevin equations in cases when the usual fluctuation–dissipation theorem does not apply. Our central result is an exact expression for the characteristic functional of the process $u(t)$, for a general d-dimensional correlated process (in appendix A we present a two-dimensional example) where the memory and the noise are arbitrary. In this way, we have been able to give a closed expression to get the whole Kolmogorov hierarchy, i.e., to calculate any n-joint probability and any n-time correlation function or cumulant.

We have applied our formalism to many different noise structures, and in this way we have shown the noise-induced interplay between the effects of the dissipative memory and the structure of the driving noises. Particular emphasis has been set in analysing the 1-time probability distribution $P(u, t)$ and its stationary state; we have shown that the transient towards the stationary state strongly depends on the interplay between the dissipation and the noise structure. As an example, we have analysed the case of an exponential memory function in the presence of different stochastic driving forces; for example, radioactive noise and Poisson noise. We have also discussed whether or not, on a large scale, a non-Gaussian distribution appears and in which case these distributions have long tails. In order to get this class of stationary distributions we have introduced, for the first time, Abel noise in connection with the occurrence of one-side power-law distributions (see appendix B). We remark that this noise structure can be useful when studying models where molecular diffusion is an important ingredient to be considered. In this context, the relaxation analysis of a generalized Langevin particle considering memory dissipation and injection of energy by microscopic random contributions characterized by a symmetric power law, can also be carried out by using the present functional formalism. In fact, asymptotically, when Abel noise is symmetric, the conclusions, for $\mu < 2$, are in agreement with the structure of the Lévy noise.

Of particular relevance is the use of our formalism to study fractional Langevin equations driven by arbitrary noise structures. We have analysed the competition between the noise statistics and the fractional derivative operator. As examples, we have compared Lévy and Gaussian noises and their interplay with the fractional calculus. We remark that the possibility of introducing arbitrary noise statistics in fractional Langevin equations is an interesting step forward to broaden the possible applications of these equations [23].

To close, let us mention that the present functional approach can also be applied to the so-called delayed Langevin equations. In that case the associated Green function $\Lambda(t)$ turns out to be the crucial ingredient for studying different models of delayed Langevin equations driven by arbitrary noises. Results along this line will be presented elsewhere.

Acknowledgment

We thank Professor V Grünfeld for the English revision.

Thus, it is simple to see that in the long-time limit ($s \sim 0$) there is map between the fractional index α and the exponent ζ that characterizes the strength of the disorder

$$\alpha = \frac{4 - 3\zeta}{4 - 2\zeta} \Rightarrow \alpha \in \left(\frac{1}{2}, 1\right].$$

Note that our functional technique also allows us to calculate more complicated objects in disordered systems; for example, we can calculate the non-trivial two-particles density $P(u_1, t_1; u_2, t_2)$, etc.

Appendix A. A two-dimensional example

In what follows we will show a multivariable example that can be fully worked out by using our formalism. It corresponds to a particle confined in a harmonic potential and driven by an arbitrary noise $\xi(t)$ (fluctuating force). The evolution for the position $x(t)$ and the velocity $v(t)$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = v(t) \tag{A1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -\Omega^2 x(t) - \int_0^t \mathrm{d}t'\,\Phi(t-t')v(t') + \xi(t) \tag{A2}$$

where Ω is the characteristic frequency of the harmonic potential. All statistical information of the processes $[x(t), v(t)]$ is encoded in the characteristic functional

$$G_{xv}([k_x(t), k_v(t)]) = \left\langle \exp\left(\mathrm{i}\int_0^\infty \mathrm{d}t\,(k_x(t)x(t) + k_v(t)v(t))\right)\right\rangle. \tag{A3}$$

For each realization of the noise, the solution of equations (A1) and (A2) reads

$$x(t) = \langle x(t)\rangle_0 + \int_0^t \mathrm{d}t'\,\mathfrak{X}(t-t')\xi(t') \tag{A4}$$

$$v(t) = \langle v(t)\rangle_0 + \int_0^t \mathrm{d}t'\,\mathfrak{F}(t-t')\xi(t'). \tag{A5}$$

Here, we have defined the averages

$$\langle x(t)\rangle_0 = x(0) + \int_0^t \mathrm{d}t'\,\langle v(t')\rangle_0 \tag{A6}$$

$$\langle v(t)\rangle_0 = v(0)\mathfrak{F}(t) - \Omega^2 x(0)\mathfrak{X}(t). \tag{A7}$$

The functions $\mathfrak{F}(t)$ and $\mathfrak{X}(t)$ are defined through their Laplace transforms

$$\mathfrak{X}(s) = \frac{\mathfrak{F}(s)}{s} \qquad \mathfrak{F}(s) = \frac{s}{s[s+\Phi(s)] + \Omega^2} \tag{A8}$$

which are equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{X}(t) = \mathfrak{F}(t) \tag{A9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{F}(t) = -\Omega^2\mathfrak{X}(t) - \int_0^t \mathrm{d}t'\,\Phi(t-t')\mathfrak{F}(t') \tag{A10}$$

with the initial conditions $\mathfrak{X}(0) = 0$ and $\mathfrak{F}(0) = 1$. After inserting equations (A4), (A5) in the definition (A3) we get the exact expression

$$G_{xv}([k_x(t), k_v(t)]) = G_{\langle x\rangle_0}([k_x(t)])G_{\langle v\rangle_0}([k_v(t)])G_\xi([z(t)]) \tag{A11}$$

where we have defined

$$G_{\langle x\rangle_0}([k_x(t)]) = \exp\left\{\mathrm{i}\int_0^\infty \mathrm{d}t\, k_x(t)\langle x(t)\rangle_0\right\} \tag{A12}$$

$$G_{\langle v\rangle_0}([k_v(t)]) = \exp\left\{\mathrm{i}\int_0^\infty \mathrm{d}t\, k_v(t)\langle v(t)\rangle_0\right\}. \tag{A13}$$

Furthermore, the scalar function $z(t)$ is defined by

$$z(t) = \int_t^\infty \mathrm{d}t' \mathfrak{X}(t'-t)k_x(t') + \int_t^\infty \mathrm{d}t' \mathfrak{F}(t'-t)k_v(t'). \tag{A14}$$

Thus, knowing the characteristic functional of the noise we can obtain the full characteristic functional of the bidimensional process $[x(t), v(t)]$.

Free Brownian motion. By taking $\Omega = 0$, the previous case reduces in a trivial way to the case of a free Brownian particle. In this case, the evolution of the particle position is given by

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -\int_0^t \mathrm{d}t' \Phi(t-t') \frac{\mathrm{d}x(t')}{\mathrm{d}t'} + \xi(t). \tag{A15}$$

This second-order differential equation also arises when modelling a rigid rotator [26, 6]. The marginal statistic description of this equation can be obtained from equation (A11) after taking $k_v(t) = 0$. Quantities like $\langle \cos[x(t)] \rangle$ or $\langle \sin[x(t)] \rangle$ follow immediately from the real and imaginary part of the characteristic functional, and more complex stochastic objects can also be calculated analytically. In general, a differential stochastic equation of any order can be analysed with our functional formalism after introducing additional stochastic process corresponding to the different derivatives of the original process.

Appendix B. Abel noise

Abel was probably the first to give an application of fractional calculus [37]. He used derivatives of arbitrary order to solve the isochrone problem in classical mechanics, and the integral equation he worked out was precisely the one Riemann used to define fractional derivatives. In a modern context, the Abel-type integral equation can be written in the form

$$\begin{aligned} \xi^{2\mu} P(\xi) &= a^\mu \frac{1}{\Gamma(\mu)} \int_0^\xi (\xi - y)^{\mu-1} P(y)\,\mathrm{d}y \qquad \mu > 0 \\ &= a^\mu {}_0 D_\xi^{-\mu} P(\xi) \end{aligned} \tag{B1}$$

where a is an arbitrary constant. It is interesting to note that the particular class of normalized one-sided Lévy-type probabilities

$$P(\xi) = \frac{a^\mu}{\Gamma(\mu)} \xi^{-1-\mu} \exp(-a/\xi) \qquad \text{with} \quad a > 0, \xi > 0 \tag{B2}$$

is a solution of the fractional differential equation (B1). The characteristic function of the probability density (B2) can be calculated taking the Laplace transform, and thus

$$\begin{aligned} G_\xi(k) &= \langle \exp(\mathrm{i}k\xi) \rangle = \mathcal{L}_s[P(\xi)]_{s=-\mathrm{i}k} \\ &= \frac{2}{\Gamma(\mu)} (\sqrt{-\mathrm{i}ka})^\mu K_\mu(2\sqrt{-\mathrm{i}ka}) \end{aligned} \tag{B3}$$

where $K_\mu(z)$ is the Basset function and $\Gamma(\mu)$ is the gamma function. Note that if $\mu \in (0, 1)$ the asymptotic behaviour of the Basset function shows the expected divergences for integer moments $\langle \xi^q \rangle$, but in general depending on the value of μ we can get finite moments

$$\langle \xi^q \rangle = a^q \frac{\Gamma(\mu - q)}{\Gamma(\mu)} \qquad \text{for} \quad \mu > q. \tag{B4}$$

We define the characteristic functional of a stochastic process $\xi(t)$, closely related to the power-law distribution (B2), in the form

$$G_\xi([k(t)]) = \frac{2}{\Gamma(\mu)} \left(\sqrt{a \int_0^\infty (1 - \mathrm{e}^{\mathrm{i}k(t)})\,\mathrm{d}t} \right)^\mu K_\mu \left(2\sqrt{a \int_0^\infty (1 - \mathrm{e}^{\mathrm{i}k(t)})\,\mathrm{d}t} \right). \tag{B5}$$

From this expression all the moments of the noise can be calculated. For example, the first moments read

$$\begin{aligned}\langle\xi(t)\rangle &= a\frac{\Gamma(\mu-1)}{\Gamma(\mu)} \qquad \text{if} \quad \mu>1\\ \langle\xi(t_1)\xi(t_2)\rangle &= a\frac{\Gamma(\mu-1)}{\Gamma(\mu)}\delta(t_1-t_2)+a^2\frac{\Gamma(\mu-2)}{\Gamma(\mu)} \qquad \text{if} \quad \mu>2.\end{aligned} \tag{B6}$$

Note that there is a constant term in the correlation function $\langle\langle\xi(t_1)\xi(t_2)\rangle\rangle \equiv \langle\xi(t_1)\xi(t_2)\rangle - \langle\xi(t_1)\rangle\langle\xi(t_2)\rangle$,

$$\langle\langle\xi(t_1)\xi(t_2)\rangle\rangle = \frac{a}{(\mu-1)}\delta(t_1-t_2)+\left(\frac{a}{(\mu-1)}\right)^2\frac{1}{(\mu-2)} \qquad \text{if} \quad \mu>2. \tag{B7}$$

As expected this constant term decreases for large values of μ. From now on we will call $\xi(t)$ Abel noise in honour of that brilliant mathematician.

Now in order to have a clear meaning of this noise we introduce an alternative interpretation. Let a stochastic process $n(t)$ be defined by a directed random walk, then for a given value of transition rate ρ we write the master equation

$$\frac{\mathrm{d}P(n,t)}{\mathrm{d}t} = \rho[P(n-1,t)-P(n,t)] \qquad \rho>0. \tag{B8}$$

Using the Darling–Siegert theorem [38] the characteristic functional of the process $n(t)$ can be found by solving *the masterly equation*

$$\frac{\mathrm{d}\Pi(n,t)}{\mathrm{d}t} = \mathrm{i}k(t)n\Pi(n,t)+\rho[\Pi(n-1,t)-\Pi(n,t)] \qquad \Pi(n,0)=\delta(n-n_0) \tag{B9}$$

where $k(t)$ is any test function. The characteristic functional of the process $n(t)$ follows from the limit

$$G_n([k(t)]) = \lim_{t\to\infty}\sum_{n=0}^{\infty}\Pi(n,t). \tag{B10}$$

Using the generating function $\Pi(Z,t) = \sum_{n=0}^{\infty} Z^n\Pi(n,t)$, the solution of the generating function of equation (B9) is, with the abbreviation $\int_0^t k(t')\,\mathrm{d}t' = K(t)$,

$$\Pi(Z,t) = (Z\exp(\mathrm{i}K(t)))^{n_0}\exp\left(\int_0^t \rho[Z\exp(\mathrm{i}K(t)-\mathrm{i}K(t'))-1]\,\mathrm{d}t'\right) \tag{B11}$$

so using equation (B10) we finally get the functional

$$G_n([k(t)]) = \exp(\mathrm{i}K(\infty)n_0)\exp\left(\rho\int_0^\infty\left[\exp\left(\mathrm{i}\int_t^\infty k(t')\,\mathrm{d}t'\right)-1\right]\mathrm{d}t\right). \tag{B12}$$

From this expression it is simple to see that

$$G_n([k(t)]) = \exp(\mathrm{i}K(\infty)n_0)G_\xi([z(t)]) \tag{B13}$$

where $G_\xi([z(t)])$ is the functional of the Poisson noise ($A=1$) evaluated in the test function $z(t) = \int_t^\infty k(t')\,\mathrm{d}t'$. This means that the Markov process $n(t)$ controlled by equation (B8), with a probability rate ρ, is equivalent to solving a linear Langevin equation driven by a Poisson noise $\xi(t)$ with a uniform density of arriving pulses ρ. Thus we can write the associated stochastic differential equation

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \xi(t). \tag{B14}$$

Now, we will assume that the rate ρ in equation (B8) is a random variable with a distribution $P(\rho)$. This assumption arises naturally in the context of disordered systems. Therefore, the final functional can be obtained from equation (B12) as

$$\langle G_n([k(t)])\rangle_{P(\rho)} = \exp(\mathrm{i}K(\infty)n_0)\int_0^\infty G_\xi([z(t)])P(\rho)\,\mathrm{d}\rho. \tag{B15}$$

This integral can be calculated for many different distributions $P(\rho)$, in particular if we use the Abel distribution (B2), we get

$$\int_0^\infty G_\xi([z(t)])P(\rho)\,\mathrm{d}\rho = \frac{2}{\Gamma(\mu)}\left(\sqrt{a\int_0^\infty(1-\exp(\mathrm{i}z(t)))\,\mathrm{d}t}\right)^\mu \\ \times K_\mu\left(2\sqrt{a\int_0^\infty(1-\exp(\mathrm{i}z(t)))\,\mathrm{d}t}\right) \tag{B16}$$

where we have used that $\mathcal{R}_e\left[\int_0^\infty(\exp(\mathrm{i}z(t)-1))\,\mathrm{d}t\right]\leqslant 0$. This result ends the interpretation of Abel noise as the noisy term appearing in equation (B14), in close connection with the solution of a directed random walk model with a random (power-law distributed) transition rate ρ.

Note that the cumulants of the Abel noise are not self-averaging with respect to the cumulants of the Poisson noise and the average over $P(\rho)$. For example, using the Poisson functional, see equation (3.19) with $A = 1$ (or equation (B12)), it is simple to show that its second cumulant is $\langle\langle\xi(t_1)\xi(t_2)\rangle\rangle = \rho\delta(t_1-t_2)$. Then the average of the correlation of the Poisson noise reads

$$\int_0^\infty \mathrm{d}\rho P(\rho)\langle\langle\xi(t_1)\xi(t_2)\rangle\rangle = a\frac{\Gamma(\mu-1)}{\Gamma(\mu)}\delta(t_1-t_2) \tag{B17}$$

a result that is different from the calculation of the correlation of the Abel noise, see equation (B7).

Consider now the subtraction of two statistically independent Abel noises $\xi^{c,b}(t)$, i.e., $\xi(t) = \xi^c(t) - \xi^b(t)$. In this case, we can write the characteristic functional of the process $\xi(t)$ in the form

$$G_\xi([k(t)]) = \left\langle\exp\left(\mathrm{i}\int_0^\infty k(t)[\xi^c(t)-\xi^b(t)]\mathrm{d}t\right)\right\rangle \\ = G_{\xi^c}([k(t)])G_{\xi^b}([-k(t)]) \tag{B18}$$

where each functional is given by equation (B5). From the properties of the Basset function we can write this formula in a compact form, using the Kelvin functions, to handle this in a more friendly way. Also from this expression, it is simple to see that the first moment of the symmetric Abel noise $\xi(t)$ is null, etc.

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